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ORIGINAL RESEARCH

A one-step implicit iterative process for a finite family of *I*-nonexpansive mappings in Kohlenbach hyperbolic spaces

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Abstract The purpose of the present paper is threefold. First, to give definition of I-nonexpansive mappings in a Kohlenbach hyperbolic space. Second, to define a new one-step implicit iterative process. Finally, to establish strong and Δ-convergence theorems for this iterative process in a Kohlenbach hyperbolic space. In addition, an example is provided to validate our result. Our results extend some existing results.

Keywords Kohlenbach hyperbolic space · I-nonexpansive map · Common fixed point · Implicit iterative process · Strong convergence · Δ-convergence

Mathematics Subject Classification 47H09 · 47H10 · 49M05

Introduction

In nonlinear functional analysis, one of the most productive tools is the fixed point theory, which has numerous applications in many quantitative disciplines such as biology, chemistry, computer science, and additionally in many branches of engineering. So, the metric fixed point theory has been investigated extensively in the past two decades by numerous mathematicians. Takahashi [1] introduced the concept of convexity in a metric space (X, d) as follows.

A convex structure in a metric space (X, d) is a mapping $W : X \times X \times [0, 1] \to X$ satisfying, for all $x, y, u \in X$ and all $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space together with a convex structure is called a convex metric space. A nonempty subset C of X is said to be convex if $W(x, y, \lambda) \in C$ for all $(x, y, \lambda) \in C \times C \times [0, 1]$.

Recently, Kohlenbach [2] enriched the concept of convex metric space by defining hyperbolic space.

A hyperbolic space [2] is a triple (X, d, W), where (X, d) is a metric space and $W : X^2 \times [0, 1] \to X$ is such that

W1. $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$
W2. $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$
W3. $W(x, y, \alpha) = W(y, x, (1 - \alpha))$
W4. $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1 - \alpha)d(z, w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$. If (X, d, W) satisfies only (W1), then it coincides with the convex metric space introduced by Takahashi [1]. All normed spaces and their subsets are the examples of hyperbolic spaces as well as convex metric spaces. It is remarked that every CAT(0) and Banach spaces are very special cases of hyperbolic space.

From now on, $\mathbb{N}$ denotes the set of natural numbers and $J = \{1, 2, \ldots, N\}$, the set of first N natural numbers. Denote by $F(T)$, the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$ and by $F := \cap_{i=1}^{N}(F(T_i) \cap F(I_i))$, the set of common fixed points of two families $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$. In what follows, we fix $x_0 \in K$ as a starting point of a process unless stated otherwise, and take $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ sequences in $(0, 1)$.

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Mann iterative process for fixed point of a mapping T is as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_{n-1}, \quad n \in \mathbb{N}. \tag{1.1}$$

In [3], Shahzad defined I-nonexpansivity of a mapping T in a Banach space. Now, we give metric version of I-nonexpansivity of a mapping T.

Let K be a nonempty subset of a metric space (X, d) and T, I be two selfmaps on K. T is said to be nonexpansive if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in K$, and T is said to be I-nonexpansive [3] if $d(Tx, Ty) \le d(Ix, Iy)$ for all $x, y \in K$.

In [4], Rhoades and Temir established the weak convergence of the sequence of the Mann iterates to a common fixed point of T and I by considering the map T to be I-nonexpansive. More precisely, they proved the following theorem.

Theorem 1 [4] *Let K be a closed convex bounded subset of uniformly convex Banach space X , which satisfies Opial's condition, and let T, I self-mappings of K with T be an I-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$,* the sequence $\{x_n\}$ of Mann iterates converges weakly to a common fixed point of $F(T) \cap F(I)$.

Temir and Gul [5] obtained weak convergence theorems of fixed points for I-nonexpansive mappings and I-asymptotically quasi-nonexpansive mappings in Hilbert space. Later on, some authors [6–8] studied convergence theorems for generalization of the class of I-nonexpansive mappings.

Concerning the common fixed points of the finite family $\{T_i : i \in J\}$, Xu and Ori [9] introduced the following implicit iterative process:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad n \in \mathbb{N} \tag{1.2}$$

where $T_n = T_{n(\mathrm{mod}\, N)}$.

Zhao et al. [10] introduced the following implicit iterative process for the same purpose.

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + \gamma_n T_n x_n, \quad n \in \mathbb{N} \tag{1.3}$$

where $T_n = T_{n(\mathrm{mod}\, N)}$.

Khan [11] generalized results of Zhao et al. [10] for two finite families of nonexpansive mappings. Plubtieng et al. [12] defined an implicit iterative process for two finite families of nonexpansive mappings $\{T_i : i \in J\}$ and $\{S_i : i \in J\}$ as follows:

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_n y_n, \\ y_n &= \beta_n x_n + (1-\beta_n)S_n x_n, \quad n \in \mathbb{N} \end{aligned} \tag{1.4}$$

where $T_n = T_{n(\mathrm{mod}\, N)}$ and $S_n = S_{n(\mathrm{mod}\, N)}$.

Khan et al. [13] studied implicit iteration (1.4) for two finite families of nonexpansive mappings in a hyperbolic space as follows:

$$\begin{aligned} x_n &= W(x_{n-1}, T_n y_n, \alpha_n), \\ y_n &= W(x_n, S_n x_n, \beta_n), \quad n \in \mathbb{N} \end{aligned} \tag{1.5}$$

Later on, some authors discussed the convergence of the iterative process in hyperbolic spaces (see, for example, [14–17]).

Motivated by the above facts, in this paper we define a new algorithm as follows:

Let K be a nonempty closed convex subset of a convex metric space X, $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings. Then $\{x_n\}$ is defined as:

$$x_n = W\left(T_n x_n, W\left(I_n x_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right) \tag{1.6}$$

where $T_n = T_{n(\mathrm{mod}\, N)}$ and $I_n = I_{n(\mathrm{mod}\, N)}$, $0 < a \le \alpha_n$, $\beta_n \le b < 1$ and satisfy $\alpha_n + \beta_n < 1$.

Obviously, (1.6) is equivalent to $x_n = \alpha_n x_{n-1} + \beta_n I_n x_{n-1} + (1-\alpha_n-\beta_n)T_n x_n$ in the Banach space setting. Thus, iteration process (1.6) is more general than the iteration process (1.1)–(1.5) and iteration process of Khan [11].

Using process (1.6), we prove some Δ and strong convergence theorems for approximating common fixed points of a finite family of I-nonexpansive mappings and a finite family of nonexpansive mappings in a uniformly convex hyperbolic space. These results improve and extend the corresponding results of Rhoades and Temir [4], Soltuz [18], Chidume and Shahzad [19], Zhao et al. [10], Khan [11]. Our results also improve the corresponding results of Plubtieng et al. [12] and Khan et al. [13] being computationally simpler.

Preliminaries

Let K be a nonempty closed convex subset of a convex metric space X, $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings. Let $\{x_n\}$ be defined by (1.6). Define a mapping $B_1 : K \to K$ by $B_1 x = W(T_1 x, W(I_1 x_0, x_0, \frac{\beta_1}{1-\alpha_1}), \alpha_1)$ for all $x \in K$. Existence of x_1 is guaranteed if B_1 has a fixed point. Now for any $x, y \in K$, we have

$$\begin{aligned} &d(B_1 x, B_1 y) \le \alpha_1 d(T_1 x, T_1 y) \\ &\quad + (1-\alpha_n) d\left(W\left(I_1 x_0, x_0, \frac{\beta_1}{1-\alpha_1}\right), W\left(I_1 x_0, x_0, \frac{\beta_1}{1-\alpha_1}\right)\right) \\ &\le \alpha_1 d(Ix, Iy) \le \alpha_1 d(x, y). \end{aligned}$$

Since $\alpha_1 < 1, B_1$ is a contraction. By Banach contraction principle, B_1 has a unique fixed point. Thus, the existence of x_1 is established. Similarly, the existence of $x_2, x_3, \ldots$ is established. Thus, the iteration process (1.6) is well defined.

A hyperbolic space (X, d, W) is said to be uniformly convex [20] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that

$$\left.\begin{array}{l} d(x,u) \leq r \\ d(y,u) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \Rightarrow d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta)r.$$

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, define a continuous functional $r(., \{x_n\}) : X \to [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by $\rho = \inf\{r(x, \{x_n\}) : x \in X\}$. The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset K of X is defined as follows:

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}) \quad \text{for any } y \in K\}.$$

The set of all asymptotic centers of $\{x_n\}$ is denoted by $A(\{x_n\})$.

It has been shown in [21] that bounded sequences have unique asymptotic center with respect to closed convex subsets in a complete and uniformly convex hyperbolic space with monotone modulus of uniform convexity.

A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ [22]. In this case, we write Δ-$\lim_n x_n = x$.

Recall that Δ-convergence coincides with weak convergence in Banach spaces with Opial's property [23].

A sequence $\{x_n\}$ in a metric space X is said to be Fej ér monotone with respect to K (a subset of X) if $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in K$ and $n \geq 1$. A map $T : K \to K$ is semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Two mappings $T, S : K \to K$ with $F \neq \emptyset$ are said to satisfy the Condition (A′) [24] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\text{either } d(x, Tx) \geq f(d(x, F)) \quad \text{or} \quad d(x, Sx) \geq f(d(x, F))$$

for all $x \in K$, where $d(x, F) = \inf\{d(x, p) : p \in F\}$.

Let $\{T_i : i \in J\}$ be a finite family of nonexpansive mappings of K with nonempty fixed points set F. Then $\{T_i : i \in J\}$ is said to satisfy the Condition (B) on K [19] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\max_{i \in J} d(x, T_i x) \geq f(d(x, F))$$

for all $x \in K$.

Khan [11] modified Condition (B) for two finite families of mappings as follows. Let $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$ be two finite families of mappings of K with nonempty fixed points set F. These families are said to satisfy Condition (B′) on K if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\text{either } \max_{i \in J} d(x, T_i x) \geq f(d(x, F)) \quad \text{or} \quad \max_{i \in J} d(x, I_i x) \geq f(d(x, F))$$

for all $x \in K$. The Condition (B′) reduces to the Condition (A′) when $T_1 = T_2 = \cdots = T_N = T$ and $S_1 = S_2 = \cdots = S_N = S$, and to the Condition (B) when $S_i = T_i$ for all $i \in J$.

For the development of our main results, some key results are listed below.

Lemma 1 [13] *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 2 [13] *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 3 [25] *Let K be a nonempty closed subset of a complete metric space (X, d) and $\{x_n\}$ be Fejér monotone with respect to K. Then $\{x_n\}$ converges to some $p \in K$ if and only if $\lim_{n\to\infty} d(x_n, K) = 0$.*

Main results

Lemma 4 *Let K be a closed and convex subset of a convex metric space X. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined by (1.6) is Fej ér monotone with respect to F.*

Proof Let $p \in F$. It follows from (1.6) that

$$\begin{aligned} d(x_n,p) &= d\left(W\left(T_nx_n, W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\ &\leq \alpha_n d(T_nx_n,p) + (1-\alpha_n) d\left(W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\leq \alpha_n d(I_nx_n,p) + \beta_n d(I_nx_{n-1},p) + (1-\alpha_n-\beta_n) d(x_{n-1},p) \\ &\leq \alpha_n d(x_n,p) + \beta_n d(x_{n-1},p) + (1-\alpha_n-\beta_n) d(x_{n-1},p) \end{aligned}$$

and this implies that

$$d(x_n,p) \leq d(x_{n-1},p).$$

Hence, $\{x_n\}$ is Fejér monotone with respect to F. □

Lemma 5 *Let K be a closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \neq \emptyset$. Then for the sequence $\{x_n\}$ in (1.6), we have*

$$\lim_{n\to\infty} d(x_n, T_kx_n) = \lim_{n\to\infty} d(x_n, I_kx_n) = 0 \quad v \text{ for each } k \in J.$$

Proof Let $p \in F$. Then $\lim_{n\to\infty} d(x_n,p)$ exists by above lemma. Suppose that $\lim_{n\to\infty} d(x_n,p) = c$. Then

$$\lim_{n\to\infty} d(x_n,p) = \lim_{n\to\infty} d\left(W\left(T_nx_n, W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) = c. \tag{3.1}$$

Since T_n is a I_n-nonexpansive mapping for all n, we have $d(T_nx_n,p) \leq d(I_nx_n,p) \leq d(x_n,p)$. Taking lim sup on both sides of this inequality, we obtain

$$\limsup_{n\to\infty} d(T_nx_n,p) \leq \limsup_{n\to\infty} d(x_n,p) = c. \tag{3.2}$$

Now

$$\begin{aligned} & d\left(W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\leq \left(\frac{\beta_n}{1-\alpha_n}\right) d(I_nx_{n-1},p) + \left(1-\frac{\beta_n}{1-\alpha_n}\right) d(x_{n-1},p) \\ &\leq \left(\frac{\beta_n}{1-\alpha_n}\right) d(x_{n-1},p) + \left(1-\frac{\beta_n}{1-\alpha_n}\right) d(x_{n-1},p) \\ &\leq d(x_{n-1},p) \end{aligned}$$

implies that

$$\limsup_{n\to\infty} d\left(W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) \leq c. \tag{3.3}$$

Using (3.1)–(3.3) and Lemma 1, we get

$$\lim_{n\to\infty} d\left(T_nx_n, W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right)\right) = 0. \tag{3.4}$$

Observe that

$$\begin{aligned} d(x_n, T_nx_n) &= d\left(W\left(T_nx_n, W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), T_nx_n\right) \\ &\leq \alpha_n d(T_nx_n, T_nx_n) + (1-\alpha_n) \\ &\quad \times d\left(W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), T_nx_n\right). \end{aligned}$$

Taking lim sup on both sides in the above inequality and using (3.4), we have

$$\lim_{n\to\infty} d(x_n, T_nx_n) = 0. \tag{3.5}$$

Next,

$$\begin{aligned} d(x_n,p) &= d\left(W\left(T_nx_n, W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\ &\leq \alpha_n d(T_nx_n,p) + (1-\alpha_n) d\left(W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\leq \alpha_n d(I_nx_n,p) + (1-\alpha_n) d\left(W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) \end{aligned}$$

yields that

$$d(x_n,p) \leq d\left(W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right).$$

Thus, we get

$$c \leq \liminf_{n\to\infty} d\left(W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right). \tag{3.6}$$

Combining (3.3) and (3.6), we get

$$\lim_{n\to\infty} d\left(W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), p\right) = c. \tag{3.7}$$

Taking lim sup on both sides of $d(I_nx_{n-1},p) \leq d(x_{n-1},p)$, we obtain

$$\limsup_{n\to\infty} d(I_nx_{n-1},p) \leq \limsup_{n\to\infty} d(x_{n-1},p) = c. \tag{3.8}$$

Now using (3.7), (3.8) and Lemma 1, we obtain

$$\lim_{n\to\infty} d(x_{n-1}, I_nx_{n-1}) = 0. \tag{3.9}$$

Consider

$$\begin{aligned} d(x_n, x_{n-1}) &= d\left(W\left(T_nx_n, W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), x_{n-1}\right) \\ &\leq \alpha_n d(T_nx_n, x_{n-1}) + (1-\alpha_n) \\ &\quad \times d\left(W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), x_{n-1}\right) \\ &\leq \alpha_n\{d(T_nx_n, x_n) + d(x_n, x_{n-1})\} \\ &\quad + (1-\alpha_n) d\left(W\left(I_nx_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), x_{n-1}\right) \\ &\leq \alpha_n\{d(T_nx_n, x_n) + d(x_n, x_{n-1})\} \\ &\quad + (1-\alpha_n)\left[\frac{\beta_n}{1-\alpha_n} d(I_nx_{n-1}, x_{n-1}) \right. \\ &\quad \left. + \left(1-\frac{\beta_n}{1-\alpha_n}\right) d(x_{n-1}, x_{n-1})\right]. \end{aligned}$$

By this inequality, we have

$$d(x_n, x_{n-1}) \leq \frac{\alpha_n}{1-\alpha_n} d(T_n x_n, x_n) + \frac{\beta_n}{1-\alpha_n} d(I_n x_{n-1}, x_{n-1}).$$

Taking limsup on both the sides in the above inequality and then using (3.5) and (3.9), we have

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = 0$$

and so

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0, \quad \text{for all } k \in J. \tag{3.10}$$

We have also,

$$\begin{aligned} d(x_n, I_n x_n) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, I_n x_{n-1}) + d(I_n x_{n-1}, I_n x_n) \\ &\leq d(x_n, x_{n-1}) + d(x_{n-1}, I_n x_{n-1}) + d(x_{n-1}, x_n) \\ &= 2d(x_n, x_{n-1}) + d(x_{n-1}, I_n x_{n-1}). \end{aligned}$$

Thus, from (3.9) and (3.10), we obtain

$$\lim_{n\to\infty} d(x_n, I_n x_n) = 0. \tag{3.11}$$

Finally, note that

$$\begin{aligned} d(x_n, I_{n+k} x_n) &\leq d(x_n, x_{n+k}) + d(x_{n+k}, I_{n+k} x_{n+k}) + d(I_{n+k} x_{n+k}, I_{n+k} x_n) \\ &\leq d(x_n, x_{n+k}) + d(x_{n+k}, I_{n+k} x_{n+k}) + d(x_{n+k}, x_n) \\ &\leq 2d(x_n, x_{n+k}) + d(x_{n+k}, I_{n+k} x_{n+k}). \end{aligned}$$

Taking lim on both sides of the above inequality, we have

$$\lim_{n\to\infty} d(x_n, I_{n+k} x_n) = 0 \quad \text{for each } k \in J.$$

Since for each $k \in J$, the sequence $\{d(x_n, I_k x_n)\}$ is a subsequence of $\bigcup_{i=1}^{N} \{d(x_n, I_{n+k} x_n)\}$ and $\lim_{n\to\infty} d(x_n, I_{n+k} x_n) = 0$ for each $k \in J$; therefore,

$$\lim_{n\to\infty} d(x_n, I_k x_n) = 0 \quad \text{for each } k \in J.$$

Similarly, we have

$$\lim_{n\to\infty} d(x_n, T_{n+k} x_n) = 0 \quad \text{for each } k \in J,$$

and hence

$$\lim_{n\to\infty} d(x_n, T_k x_n) = 0 \quad \text{for each } k \in J.$$

□

Firstly, we state a result concerning Δ-convergence for algorithm (1.6). The method of proof closely follows [13, Theorem 3.1].

Theorem 2 *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (1.6), Δ-converges to a common fixed point of $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$.*

Proof It follows from Lemma 4 that $\{x_n\}$ is bounded. Therefore, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Assume that $\{u_n\}$ is any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by Lemma 5, we have $\lim_{n\to\infty} d(u_n, T_k u_n) = \lim_{n\to\infty} d(u_n, I_k u_n) = 0$ for each $k = 1, 2, \ldots, N$. Now we prove that u is the common fixed point of $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$.

Define a sequence $\{v_m\}$ in K by $v_m = T_m u$, where $T_m = T_{m(\mathrm{mod}\, N)}$. Clearly,

$$\begin{aligned} d(v_m, u_n) &\leq d(T_m u, T_m u_n) + d(T_m u_n, T_{m-1} u_n) + \cdots + d(T u_n, u_n) \\ &\leq d(u, u_n) + \sum_{i=1}^{m-1} d(u_n, T_i u_n). \end{aligned}$$

Thus, we have

$$r(v_m, \{u_n\}) = \limsup_{n\to\infty} d(v_m, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(v_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. By Lemma 2, we obtain $T_{m(\mathrm{mod}\, N)} u = u$, which implies that u is the common fixed point of $\{T_i : i \in J\}$. Similarly, we can show that u is the common fixed point of $\{I_i : i \in J\}$. Therefore, $u \in F$. Moreover, $\lim_{n\to\infty} d(x_n, u)$ exists by Lemma 4.

Assume $x \neq u$. By the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

a contradiction. Thus, $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$; therefore, $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to a common fixed point of $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$. □

Secondly, we give strong convergence theorems of algorithm (1.6).

Theorem 3 *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (1.6) converges strongly to $p \in F$ if and only if* $\lim_{n\to\infty} d(x_n, F) = 0$.

Proof It follows from Lemma 4 that $\{x_n\}$ is Fejér monotone with respect to F and $\lim_{n\to\infty} d(x_n, F)$ exists. Now applying the Lemma 3, we obtain the result. □

Theorem 4 *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \neq \emptyset$. Suppose that $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$ satisfy condition (B'). Then the sequence $\{x_n\}$ defined in (1.6) converges strongly to $p \in F$.*

Proof By Lemma 4, $\lim_{n\to\infty} d(x_n, F)$ exists for all $p \in F$. In addition, by Lemma 5, $\lim_{n\to\infty} d(x_n, T_k x_n) = \lim_{n\to\infty} d(x_n, I_k x_n) = 0$ for each $k \in J$. It follows from condition (B′) that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing with $f(0) = 0$, we have $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Theorem 3 implies that $\{x_n\}$ converges strongly to a point p in F. □

Note that the Condition (B′) is weaker than both the compactness of K and the semi-compactness of the nonexpansive mappings $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$ (see [26]); therefore, we already have the following result.

Theorem 5 *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i : i \in J\}$ be a finite family of I_i-nonexpansive mappings and $\{I_i : i \in J\}$ be a finite family of nonexpansive mappings on K such that $F \neq \emptyset$. Suppose that either K is compact or one of the map in $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.6) converges strongly to $p \in F$.*

Proof Use Lemma 5 and the line of action given in the proof of Theorem 3 in [11]. □

By taking $T_i = T$, $I_i = I$ for all $i \in J$ in the Theorem 2–5, we get the following corollary, yet is new in itself.

Theorem 6 *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let T be a I-nonexpansive mapping and I be a nonexpansive mapping on K such that $F = F(T) \cap F(I) \neq \emptyset$. Let the sequence $\{x_n\}$ defined by*

$$x_n = W\left(Tx_n, W\left(Ix_{n-1}, x_{n-1}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right) \tag{3.12}$$

(i) *Then $\{x_n\}$ Δ-converges to a common fixed point of T and I.*
(ii) *Then $\{x_n\}$ converges strongly to $p \in F$ if and only if $\lim_{n\to\infty} d(x_n, F) = 0$.*
(iii) *If T and I satisfy condition (A'), then $\{x_n\}$converges strongly to a point in F.*
(iv) *If either K is compact or one of the mappins in TandIis semi-compact, then $\{x_n\}$ converges strongly to $p \in F$.*

To testify our above theorem, we give the following numerical example.

Example 1 Let $X = (-\infty, +\infty)$ with the usual norm $\|$ and $K = [0, 1]$. In this case, (1.6) reduces to $x_n = \alpha_n x_{n-1} + \beta_n I x_{n-1} + (1 - \alpha_n - \beta_n) T x_n$. Define $T, I : K \to K$ as $Tx = \frac{2x+1}{4}$ and $Ix = 1 - x$. Then one can see that T is a I-nonexpansive mapping and I is a nonexpansive mapping on K with common fixed point set $\{\frac{1}{2}\}$. Also K *is compact and* T,I are semi-compact. *Set* $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{1}{n+2}$. *Thus, all assumptions of above theorem are satisfied.* Now we show that $\{x_n\}$ converges strongly to $\frac{1}{2}$. By taking $n = 1$ and $x_0 \in K$, we get $\alpha_1 = \frac{1}{2}, \beta_1 = \frac{1}{3}, 1 - \alpha_1 - \beta_1 = \frac{1}{6}$ and find x_1 from $x_1 = \alpha_1 x_0 + \beta_1 I x_0 + (1 - \alpha_1 - \beta_1) T x_1$. Similarly, $x_2, x_3, \ldots, x_n, \ldots$. We obtain the first ten terms of $\{x_n\}$ as in following table for initial value $x_0 = 0$, $x_0 = 0.2$, $x_0 = 0.7$ and $x_0 = 1$, respectively. From the table below, we see that the sequence $\{x_n\}$ converges strongly to $\frac{1}{2}$. This means that above theorem is applicable.

Iteration no. n	$x_0 = 0$ x_n	$x_0 = 0.2$ x_n	$x_0 = 0.7$ x_n	$x_0 = 1$ x_n
1	0.4090909091	0.4454545455	0.5363636364	0.5909090909
2	0.4904306220	0.4942583733	0.5038277512	0.5095693780
3	0.4993400429	0.4996040258	0.5002639829	0.5006599571
4	0.4999678070	0.4999806842	0.5000128772	0.5000321930
5	0.4999988293	0.4999992976	0.5000004682	0.5000011707
6	0.4999999670	0.4999999803	0.5000000132	0.5000000330
7	0.4999999993	0.4999999995	0.5000000002	0.5000000007
8	0.5000000000	0.5000000000	0.5000000000	0.5000000000
9	0.5000000000	0.5000000000	0.5000000000	0.5000000000
10	0.5000000000	0.5000000000	0.5000000000	0.5000000000

Remark 1

(1) Our results generalize the corresponding results in [10, 11] for more general class of I-nonexpansive maps in the general setup of hyperbolic spaces.
(2) Theorem 2 gives an analogue of Khan's weak convergence result [11] for a finite family of I_i-nonexpansive maps on unbounded domain in a uniformly convex hyperbolic space X.
(3) In view of simplicity of the iterative process (1.6) as compared with (1.4), our results improve and generalize the results of Khan et al. [13].

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